CABRI® II Plus

Innovative Maths Tools

ADVANCED

Welcome to the **Advanced** part of Cabri II Plus user manual.

This 3 chapter section presents some advanced problems amusing to explore and easy to solve using Cabri geometry. Those problems complement the **Tutorial** section for users who intend to pursue their discovery of Cabri II Plus.

Those exercises are designed for advanced level or undergraduate work. They are largely independent from each other and the reader is invited to duplicate the detailed construction methods and then try the listed exercises. Exercises marked with an asterisk are more difficult.

# CONTENTS

MOVING ON

CHAPTER 1

PEDAL TRIANGLES

Use the [Points]Point ⊡ tool to start with three points, *A, B, C,* anywhere on the drawing area. First construct the straight lines *AB, BC* and *CA,* using the [Lines]Line ⊞ tool. Create a fourth point *M,* anywhere on the plane, and the orthogonal projections of *M : C', A'* and *B',* respectively, on these lines. These points are constructed by first creating the perpendiculars through *M* to each of the lines in turn, using the [Constructions]Perpendicular Line ⊞ tool. Use the [Points]Point ⊡ tool to pick up the point of intersection of each perpendicular with its corresponding line. The [Points]Point ⊡ tool constructs implicitly the points of intersection of two objects. It merely requires the cursor to be placed close to an intersection, when Cabri II Plus displays the message Point at this intersection or, in an ambiguous case Intersection of... followed by a menu list.

The three points *A', B'* and *C'* define a triangle which can be drawn using the [Lines]Triangle △ tool. It is called the pedal triangle of *ABC.* The interior of the triangle can be coloured, using the [Attributes]Fill... ⊞ tool. The point of interest here is the area of the triangle with regard to the position of M. The area of the triangle is measured, using the [Measurements]Area ⊞ tool. The resulting value is a «geometrical» area, taking no account of the orientation of the triangle. The measurement is given in cm² and can be placed anywhere on the drawing area. By clicking on the number with the right mouse button, a shortcut menu appears, with the option to change to the «algebraic» area, the sign of which depends on the orientation of the triangle.
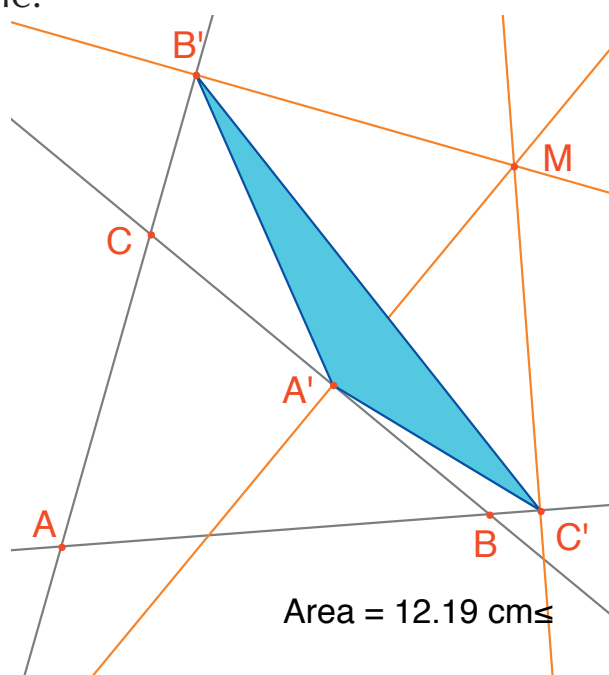

Area = 12.19 cm≤

Figure 1.1 - The pedal triangle for M, *and its area.*

We shall consider how the area of *A'B'C'* varies, depending on the position of *M*. There are several possible strategies for this. For example, activate the [Text and Symbols]Trace On/Off ![icon] tool (which requires selection of the object to be traced, *M* here – so click on it). Now move M while attempting to keep the area of *A'B'C'* constant. Successive positions of *M* are displayed on the screen, giving the general appearance of a contour line for equal values of the area of *A'B'C'*. Another strategy could be to use the locus of points on a grid to draw a visual representation of the area of *A'B'C'* for a large number of positions of *M*.

Here, we shall use this latter strategy, and draw the circle, center *M*, which has an area proportional to that of *A'B'C'* for a large number of positions of *M*. To do this, it is necessary first to calculate the radius of the circle, proportional to the square root of the area of the triangle. Activate the [Measurements]Calculate... ![icon] tool, and enter the expression sqrt(then select the number displaying the area of the triangle) to insert that into the expression, which becomes sqrt(a. Now close the bracket). Divide by 10 to avoid having a circle which is too large. The expression in the calculator is now sqrt(a)/10. Evaluate this by clicking on the = button, then drag the answer to an appropriate position on the sheet. To draw a circle, centre *M*, using the radius we have just calculated, activate the tool [Constructions]Compass ![icon]. Select the number, just moved onto the sheet, then the point *M*. The circle, center *M*, with the required radius appears. We can now see the changes in the area of the circle surrounding *M*, as the point is moved.
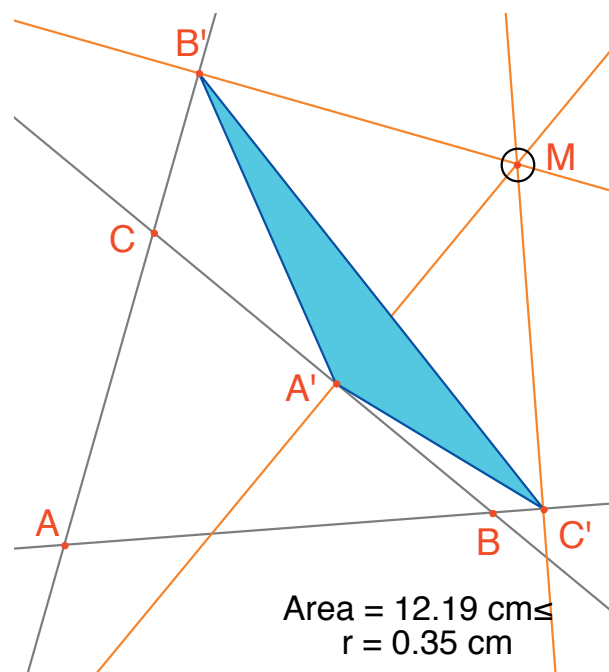

Figure 1.2 - *A circle is drawn, centre* M, *with area proportional to that of* A'B'C'.

We shall now define a grid, and redefine *M* in terms of the grid, then draw the circles representing the area of the pedal triangle at each point of the grid. To define the grid, a system of axes is required. We shall take the default axes which are available for any figure. To display them choose [Attributes]Show Axes ⬛. Next, activate the [Attributes]Define Grid ⬛ tool, and select the axes. A grid of points appears.
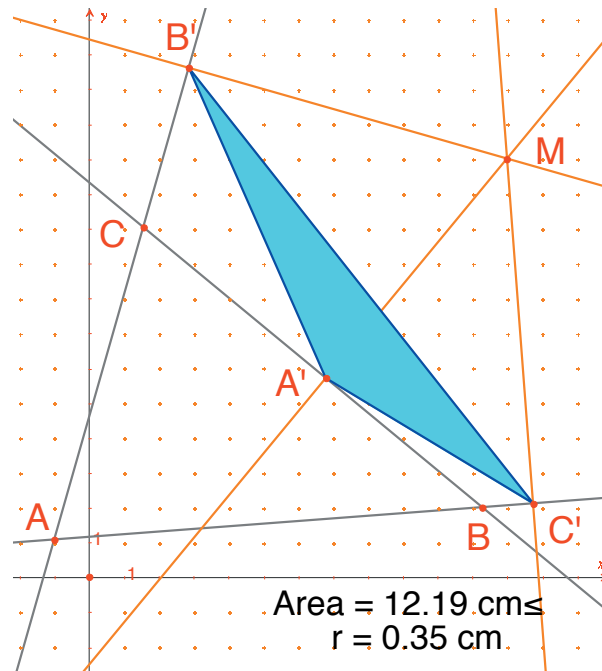

Figure 1.3 - A grid is constructed, using the default axes for the figure. M *is then redefined as any point on the grid.*

M is still an independent, movable point in the plane; we shall redefine it so that it is limited to the grid points. Activate the [Constructions]Redefine Object ⬛ tool, then select *M*. Choose the option Point on an object from the menu list that appears, and then select any point on the grid. *M* is now constrained to the points of the grid.

The [Constructions]Locus ⬛ tool can now be used to construct the set of circles which are obtained by moving *M* around the grid. Select the circle then the point *M* to obtain the locus of circles as *M* moves over the grid.

It can be shown (see for example *Geometry* Revisited by *H.M.S. Coxeter and S.L. Greitzer, Mathematical Association of America, section 1.9*) that the contour lines of equal areas of the pedal triangles are circles with the same centre as that of the circumcircle of *ABC*. In particular, triangle *A′B′C′* has zero area if *M* is on the circumcircle of *ABC*, or equivalently, points *A′, B′* and *C′* are collinear if and only if *M* lies on the circumcircle of *ABC*.
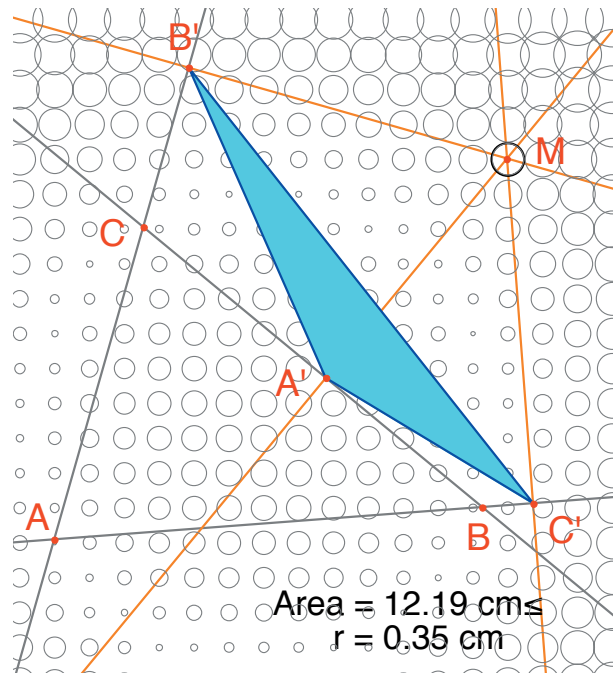
Figure 1.4 - *The distribution of the area of the pedal triangle as a function of the position of* M.

Exercise 1 - With *M* on the circumcircle of *ABC*, the three points *A', B'* and *C'* are collinear and *A'B'C'* is called the *Simson* [1] line for *M* (or *Wallace* [2] line – this line was incorrectly attributed to *Simson* for many years, as it was in fact published in 1799 by *Wallace*). Construct the envelope of *Simson* lines. (Use the [Constructions]Locus 📈 tool). This curve, which is invariant under a rotation through 120°, is called a deltoid (or tricuspoid or *Steiner's* [3] hypocycloid), since its shape is rather like that of the Greek letter Δ. It is tangent to the three lines *AB, BC* and *CA*. It is an algebraic curve of degree 4. You can check this in asking for its equation through the [Measurement]Equation or Coordinates [(x,y) y=f(x)] tool.

Exercise 2 - For the deltoid of the previous exercise, construct the center, the three points where the curve touches the three straight lines, and the largest circle which can be inscribed in the curve.

[1] *Robert Simson, 1687-1768*
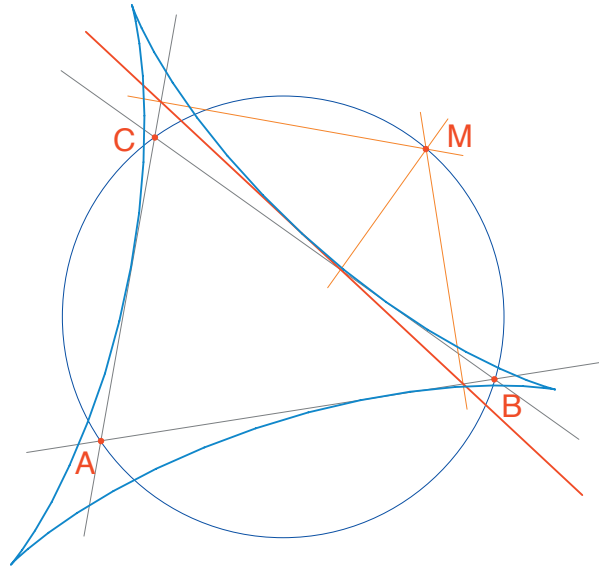[2] *William Wallace, 1768-1843*
[3] *Jakob Steiner, 1796-1863*

Figure 1.5 - *The envelope of the* Simson *lines of triangle* ABC *is called a deltoid. It has the same symmetries as an equilateral triangle.*

CHAPTER 2

Graphs of functions are easy to construct in Cabri II Plus, thanks to its system of axes and the expressions tool. The graph can then be used to study the properties of the function. In this chapter, we shall study the polynomial function of degree 3.

$$f(x) = x^3 - 2x + \frac{1}{2}$$

First, display the coordinate axes, using [Attributes]Show Axes ➡. Next, we need to create the corresponding expression on the drawing area. Once an expression has been placed on the drawing area, its value can be calculated for different values of its variables. For this function, activate [Text and Symbols] Expression $\frac{3x+}{2y}$, and type in *x³ - 2*x + 1/2*. The permitted names for variables are the letters: *a, b, c... z*. Mark a point *P*, somewhere on the *x*-axis (using the [Points]Point • tool. Display its coordinates by activating [Measurement]Equation or Coordinates $\frac{(x,y)}{y=f(x)}$, then selecting *P*. The text displaying the coordinates is initially attached to *P*, and moves with the point. Using the [Manipulation]Pointer ⬉ tool, the coordinates can be detached from *P*, and placed anywhere on the figure. To return them to the point, click-and-drag close to *P*.
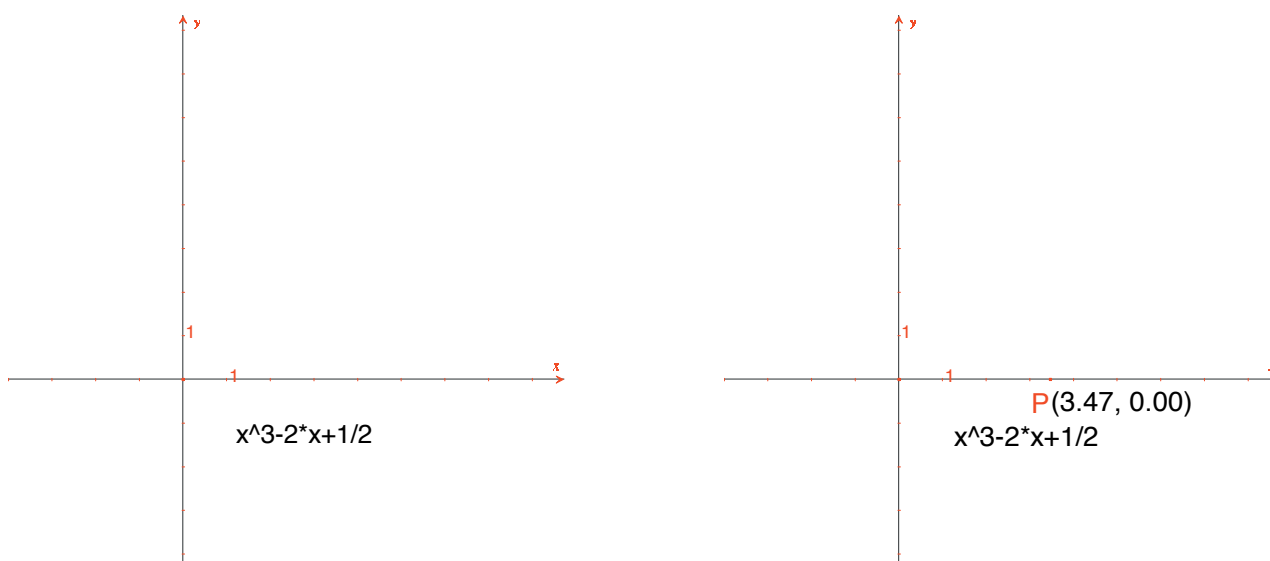
x^3-2*x+1/2

P(3.47, 0.00)

x^3-2*x+1/2

Figure 2.1 - [Left]. The expression corresponding to the function is entered on the figure. [Right]. Point P is marked on the x-axis and its coordinates displayed using [Measurement]Equation or Coordinates.

Next, we need the value of *f(x)* when x is the x-coordinate of *P*. Activate the [Measurement]Apply an Expression ![3x+2y=] tool, and click on the expression, then the x-coordinate of *P* in the brackets. Here, the order is important.
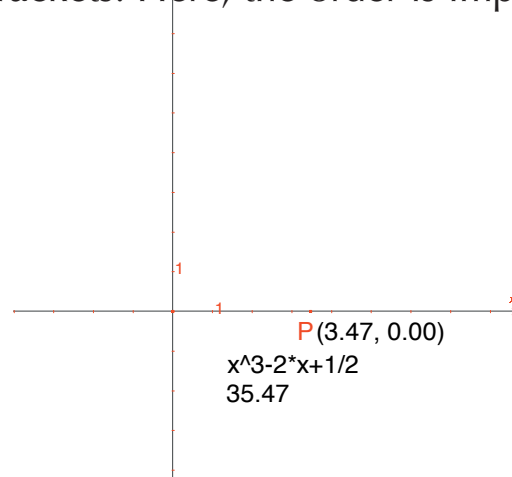
P (3.47, 0.00)
x^3-2*x+1/2
35.47

Figure 2.2 - The [Measurement] Apply an Expression tool is used to calculate the value of f(x) at the x-coordinate of P.

This value is now transferred to the *y*-axis, using the [Constructions]Measurement Transfer ![2.1] tool, and then selecting the value followed by the *y*-axis. After this, one merely has to construct the lines parallel to each of the axes, through each of the marked points, using the [Lines]Parallel Line ![icon] tool. Their point of intersection can be labelled *M*, and has coordinates *(x,f(x))*. In the following figure we have moved *P* to a point closer to the origin, (1.89,0) so that *M* is visible on the sheet.

M

0.5

0.5

P (1.89, 0.00)
x^3-2*x+1/2
3.45
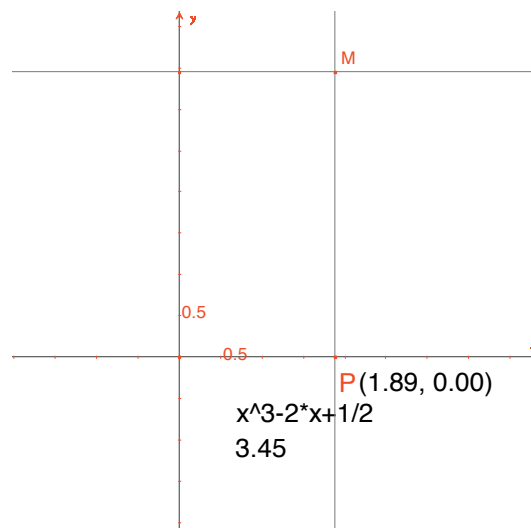
Figure 2.3 - Construction of the point M(x,f(x)) using measurement transfer.

The graph of the function is obtained as the locus of *M* as *P* moves along the *x*-axis. It is constructed using the [Constructions]Locus ![icon] tool by selecting *M* then *P*. In order to see the interesting part of the graph of the function, the origin can be moved (using drag-and-drop), and the scale changed (by dragging-and-dropping any of the scale marks on the axis).
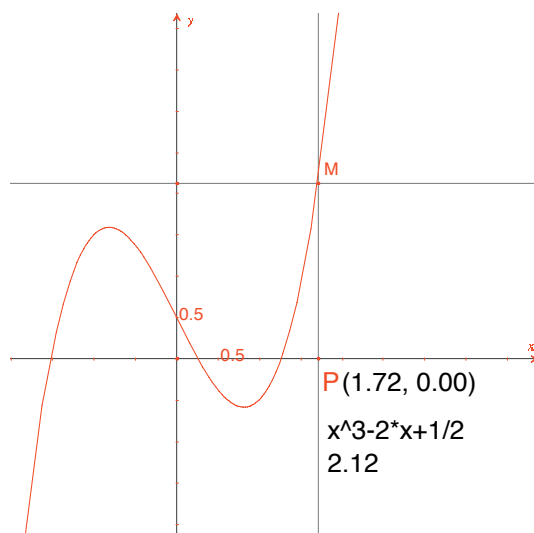
Figure 2.4 - *The graph of the function is finally created using the tool* [Construction]Locus.
The system of axes can be moved and resized so that the interesting part can be seen.

We shall now construct an approximation of the tangent to this curve at a given point. For small values of h, it is known that

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

From the geometrical point of view, this approximation takes the gradient of the tangent to be the same as the gradient of the chord linking the points on the curve whose x-coordinates are $x- h$ and $x+h$. Using [Text and Symbols]Numerical Edit **2.1[** , a value for h is defined, 0.3 here for ease of construction. The value of h can then be changed to a smaller one, giving a better approximation to the tangent. Next, construct a point A on the x-axis, and the circle center A, radius h.

The circle is obtained by activating the [Constructions]Compass ⊙ tool then selecting the value h followed by point A. The two points of intersection of this circle with the x-axis have x-coordinates $x-h$ and $x+h$, if x is the x-coordinate of A. Draw the three lines parallel to the y-axis ([Constructions]Parallel Line ⬌) which pass through the two points of intersection, and the point A. The points of intersection of these three lines with the curve provide the points B^-, B, B^+ which are points on the curve with x-coordinates $x-h$, x, and $x+h$, respectively.

As the figure is becoming rather cluttered, hide those elements which are no longer being used. Activate the [Attributes]Hide/Show ⇄ tool, and select the elements to be hidden. Here, we should hide P, M, the two construction lines for M, the coordinates of P, and the value of the function at P. The hidden objects can only be seen as marquees («marching ants» outlines), and are only visible when the [Attributes]Hide/Show ⇄ tool is active. In order to make a hidden object visible once more; just reselect it when this tool is active.
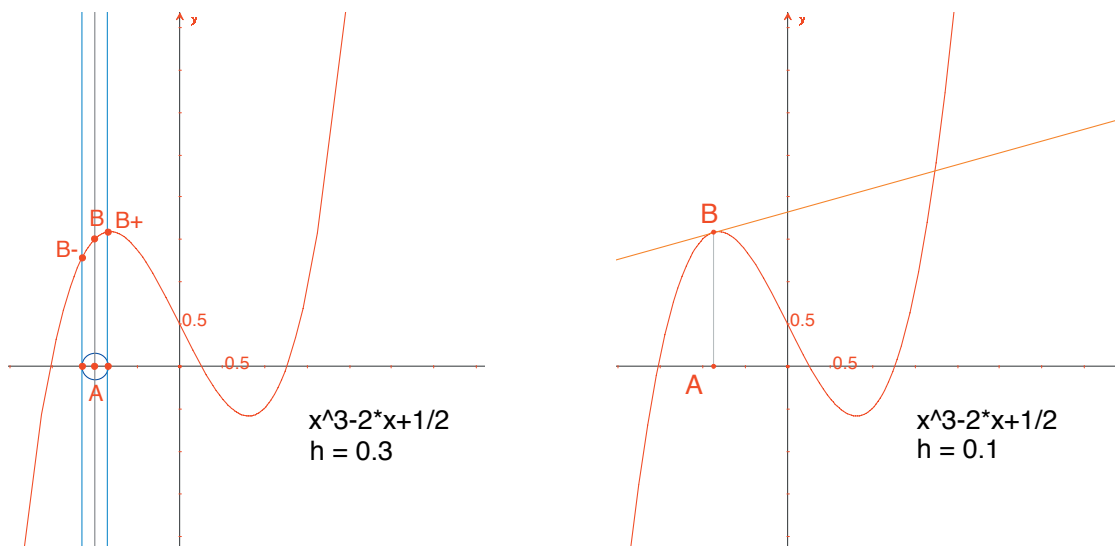

Figure 2.5 - *[Left].The three points on the curve* B⁻, B, B⁺ *with* x-*coordinates* x- h, x, *and* x+h *are constructed.*
[Right]. The approximation to the tangent at B, *once the construction elements have been hidden.*

The approximation to the tangent is now the line parallel to B^-B^+ which passes through B. Construct the latter line using the [Lines]Line ⟋ tool, then the line parallel to it using [Constructions]Parallel Line ⟋. Now hide the line through B^-B^+ and the other construction elements until only h, A, B and the «tangent» at B are visible. It can be seen that the value $h= 0.3$ already gives a very good approximation to the tangent. Nevertheless, this can be improved by decreasing the size of h, for example by taking 0.0001.

By moving the point A along the x-axis, it is possible to see the position of the three roots of the equation $f(x) = 0$, the stationary points of f, and the point of inflexion of the curve.

For information, the three solutions of $f(x) = 0$ are approximately $r_1 = -1{,}52568$, $r_2 = 0{,}25865$ and $r_3 = 1{,}26703$. The x-coordinates of the stationary points are $e_1 = -\sqrt{6}/3 \approx -0{,}81649$ and $e_2 = \sqrt{6}/3 \approx 0{,}81649$. The point of inflexion is at $(0, 1/2)$.

Exercise 3 - Using the gradient of the tangent, draw the graph which approximates the curve of the gradient function.

Exercise 4 - The tangent cuts the *x*-axis at a point *A'* with *x*-coordinate *x'*, which is, in general, a better approximation to the root, provided *A* is already in the neighborhood of a root of *f(x) = 0*. This statement is the basis of the iterative method known as the *Newton[1] - Raphson[2]* method for finding the root of an equation. Construct the point *A'*, then iterate *A"* by the same method, and compare the position of *A"* to that of *A*. In particular, two positions can be found for *A*, other than the three roots, for which *A"* and A coincide. For information, these are the two real roots of a polynomial of degree 6, whose values are approximately - 0.56293 and 0.73727. It can also be seen that a poor choice of A can cause the method to diverge, by choosing *A* so that *A'* is one of the two points where the derivative is zero.
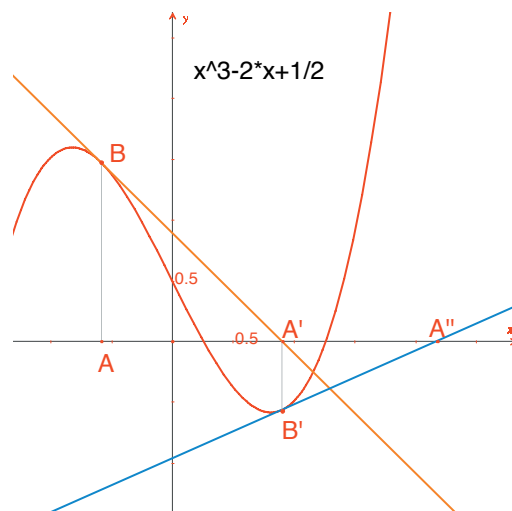

Figure 2.6 - The first two iterations of the Newton -Raphson *method, starting from point* A.

Note: the same graph can be obtained straightforwardly through the [Measurement] Apply an Expression ⬚ tool.

[1] *Sir Isaac Newton, 1643-1727*
[2] *Joseph Raphson, 1648-1715*

CHAPTER 3

We shall construct several tessellations of the plane, using polygons. Let us start with some simplified definitions, which are sufficient for the following work. An interested reader might refer to *Tilings* and *Patterns* by *Branko Grünbaum and G.C. Shepherd, Freeman 1987.* A large number of Internet websites also give information about tessellations and symmetry groups.

We say that a set of closed plane shapes is a tessellation of the plane if their interior parts are non-overlapping, and the union of all the enclosed parts covers the entire plane. These plane shapes are called tiles of the tessellation. The intersection of two tiles which is a segment of a line or a curve is called an edge, and the intersection of two or more tiles at a single point is called a vertex.

For the tessellation P, we write $S(P)$ for the set of isometries, f, of the plane such that the image of every tile of P under f is a tile of P. $S(P)$ is a group, called the symmetry group of the tessellation. There are several cases to be considered for such a group:

• $S(P)$ contains no translations. $S(P)$ is then isomorphic to a cyclic group (possibly reduced to the identity element) generated by rotation through $2\frac{1}{2}/n$, or to a dihedral group, being the symmetry group of a regular polygon with n sides.

• $S(P)$ contains translations which are all collinear. $S(P)$ is then isomorphic to one of the seven frieze groups.

• $S(P)$ contains two translations vectors which are non-collinear. Then $S(P)$ is isomorphic to one of the 17 wallpaper groups (or plane crystallographic groups), and the tessellation is said to be periodic.

If all the tiles of the tessellation can be obtained as isometries of a single tile, we say that the tessellation is monohedral.In this part, we are only interested in the case of monohedral tessellations by tiles which are polygons. We shall first construct a monohedral tessellation of the plane by a triangle.

Construct a general triangle *ABC*, using the [Lines]Triangle ◺ tool, then the midpoint, *I*, of one of its sides, *BC* for example, using the [Constructions]Midpoint ⬌ tool. Let *D* be the image of *A* under a half-turn about *I* (point symmetry), which is created using the [Transformations]Symmetry ⬤•• tool, selecting first the object to be transformed: *A*, then the center: *I*.
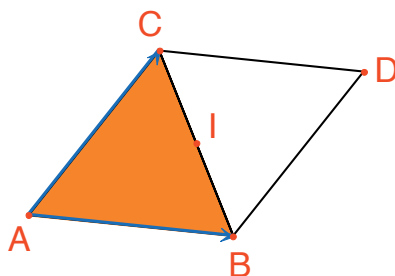

Figure 3.1 - *The image of triangle ABC is created, under rotation through 180°, about the midpoint of one of its sides (⟦BC⟧ here). This produces a parallelogram ABDC.*

The quadrilateral *ABDC* is a parallelogram and it can be used to tessellate the plane. The two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are created next, using the [Lines]Vector ➝ tool. Those vectors are used to duplicate the triangles *ABC* and *BCD* using the [Transformations]Translation ➚• tool.
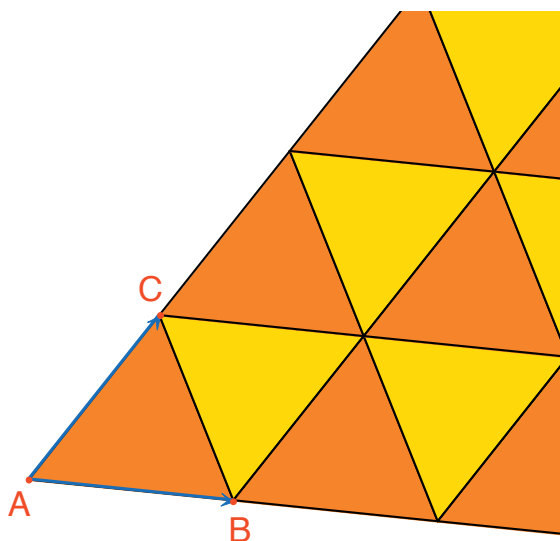

Figure 3.2 - *The [Transformations] Translation tool is used to create the images of the two triangles under translation by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.*

The same approach can be used to tessellate the plane with any quadrilateral, convex or otherwise but not crossing sides. The image of the quadrilateral is created under rotation about the midpoint of one of its sides. This produces a hexagon whose sides are pairwise parallel, which is then used to tessellate the plane by translations.
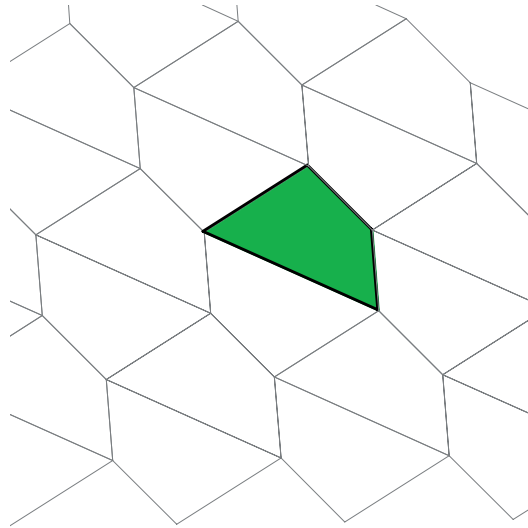
Figure 3.3 - *The same type of construction is used to tessellate the plane with any quadrilateral, convex or concave, provided it is not a crossed quadrilateral.*

For other convex polygons, the situation is much more complex. It can be shown that it is impossible to tessellate the plane with a convex polygon with more than 6 sides. There are three types of convex hexagon which will tessellate the plane and at least 14 types of convex pentagon, each type being defined by a set of constraints on the angles and the sides. At the present time, it is still not known if the 14 known types constitute the complete solution to the problem. The last of the 14 was discovered in 1985. As far as we know, the question of concave polygons has not been resolved.

Exercise 5 - Construct a convex pentagon *ABCDE*, subject to the following constraints: the angle at *A* is 60°, at *C* it is 120°, *AB* = *AE*, *CB* = *CD*. These constraints do not define a unique pentagon, but a family of pentagons. For the construction there are at least three independent points.
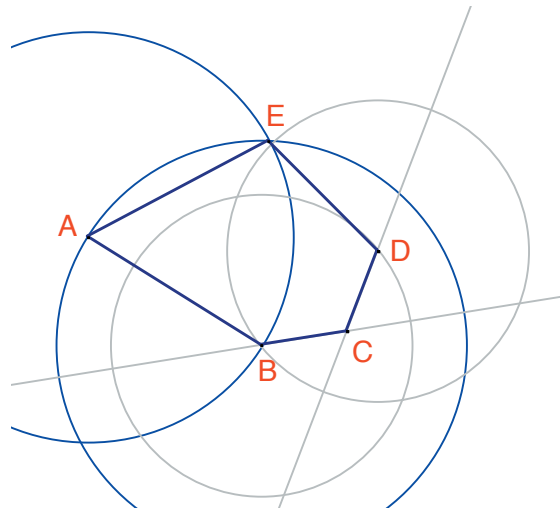

Figure 3.4 - *Construction of a pentagon under the constraints:* Â = *60°,* Ĉ = *120°,* AB = AE *and* CB = BD. A, B *and* C *are independent points in the plane.*

Make successive rotations about A through an angle of 60° using the [Transformations]Rotation ![icon] tool. This tool requires selection of: the object to be transformed, an angle, and the center of rotation, to construct a «flower» with 6 pentagonal petals. The angle required by the tool is a number on the drawing area, which has previously been created using the [Text and Symbols]Numerical Edit 2.1 tool.
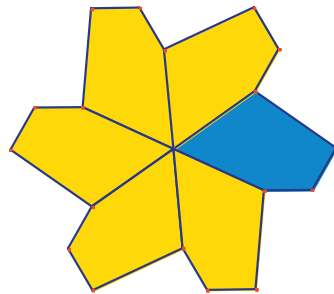

Figure 3.5 - *The basic pentagon is duplicated by rotation about center A, through an angle of 60°, to form a 6-petalled «flower».*

These flowers can now be assembled, using translations, to tessellate the plane. This tessellation is type 5 according to the classification given in *Tilings and Patterns*. It was first published by *K. Reinhardt* in 1918. This tessellation is not only monohedral, that is to say that all the tiles are identical within an isometry, but it is also isohedral: all pentagons are surrounded by the same pattern of pentagons in the tessellation.
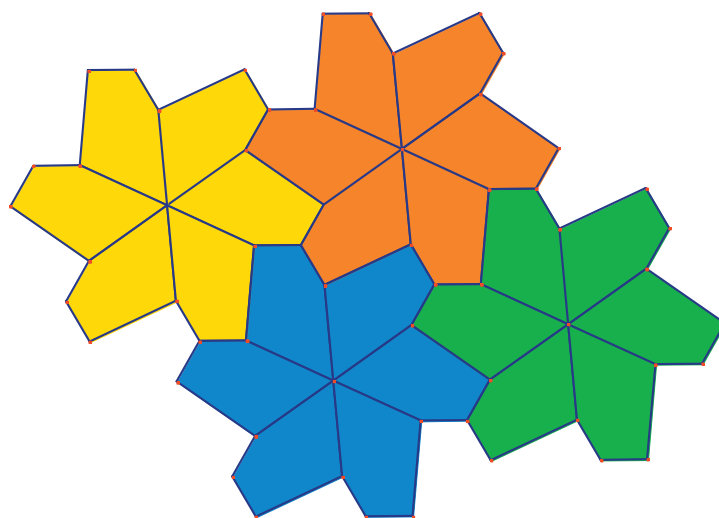

Figure 3.6 - *The flowers are assembled by translations to cover the plane.*

Exercise 6 - Construct a Pentagon *ABCDE* with constraints:
$\hat{E} = 90°$, $\hat{A} + \hat{D} = 180°$, $2\hat{B} - \hat{D} = 180°$, $\hat{C} + \hat{D} = 360°$, $EA = ED = AB + CD$.
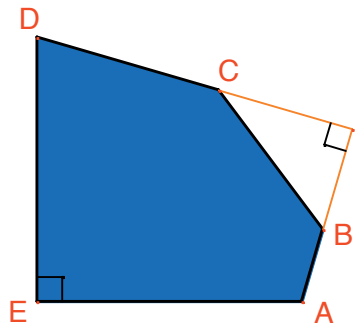

Figure 3. 7 - A pentagon of type 10, according to the classificationin Tilings *and* Patterns. *This pentagon is the basis for a monohedral tessellation of the plane. Points* A *and* E *are independent points of the plane, and point* I *is free to move on the arc of a circle.*

The tessellation is constructed by first making three copies of the tile, using successive rotations through 90° about *E*, to obtain a truncated square. These squares are then assembled in strips using translation in one direction. The strips of squares are then separated by strips of pentagons, as shown below.
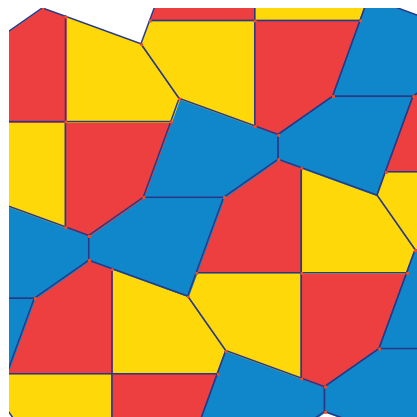

Figure 3.8 - A monohedral tessellation of the plane by convex pentagons. This tessellation was created by Richard E. James III, *following the publication of an article by* Martin Gardner *in* Scientific American *in 1975. The complete article can be found in «*Time travel and other mathematical bewilderments»*,* Martin Gardner, *Freeman 1987.*