CABRI® II Plus

Innovative Math Tools

TESSELLATIONS OF PENROSE

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This chapter makes reference to the definitions given in the preceding chapter **TESSELLATIONS** of the advanced problem manual. There are sets of polygons which cannot be used to produce periodic tessellations. The best known of these are the *Penrose* tessellations, named after the mathematician *Roger Penrose[1]* who discovered them in 1974. These tiles are called *Kites and Darts*. A coloured motif is drawn on the tiles, and only those assemblies that respect the correspondence of the colours are allowed. This eliminates all the periodic tessellations. These two tiles are quadrilaterals whose angle are multiples of $\emptyset = 36°$, and whose sides are of length 1 and $\emptyset$ (phi), where $\emptyset = (1 + \sqrt{5})/2$ the golden ratio. The colour motif shown here was drawn by *John Conway[2]*, and gives astonishing curves that are invariant under rotation through $\emptyset$.
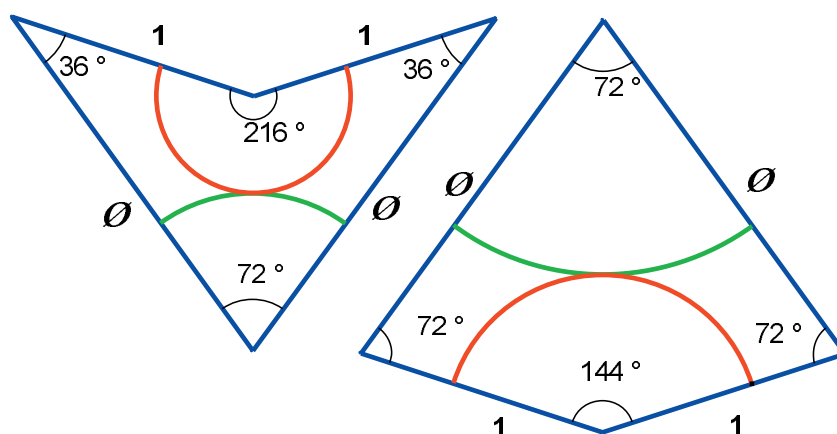

Figure 1 - A dart tile (left) and a kite tile (right).

The *Kite* and *Dart* tiles have lengthy constructions, so we will create a construction macro which enables copies of these tiles to be deposited liberally across the drawing sheet by means of a single click.

A (construction) macro is created from a sub-section of a figure. Its definition is based on a set of initial objects, and a set of final objects which are constructed uniquely from the initial objects. Once the macro has been defined, it is available as a new tool in the [Macros] toolbox. To complete the construction, the user has to select a set of objects, of the same type as the initial objects, and the tool reproduces the construction stored in the macro, using these selected objects.

[1] *Sir Roger Penrose, 1931*
[2] *John Horton Conway, 1937*

When the macro is created, it is possible to name it, to design an icon for it, and to save it in a separate file. To use a macro created for another figure, open that figure and the macro will be available for all opened figures. The macro is saved in a figure file if it is either used or created in that figure.

It is possible to redefine a macro, by defining a macro which has the same name, and constructs objects of the same type. When the user attempts to save the macro, Cabri II Plus asks whether it should replace the previous macro, or supplement it. If the latter choice is made, either of the macros can be used. For example, a macro can be defined which starts either from two points or from a line segment.

We will define a macro named `Dart 1 L`, which starts from two points, *A* and B, and constructs a `Dart tile` on *AB*, to the right of it when looking at *B* from *A*, with *AB* as a short side of the tile (hence the `1` in the name) and an arc, one foot of which is further from *A* than from *B*(hence the `L` for long). In similar fashion we will define the macros `Dart 1 S` (short) whose shape is the same as that of `Dart 1 L`, but where the foot of the arc is nearer to *A* than to *B*. `Dart phi L`, `Dart phi S` need to be constructed in the same way, as well as the four corresponding macros for the Kite.

To define these macros, we first construct the tiles, starting from two points. So, let us take any two points, *A* and *B*, constructed with the [Points]Point ⟨•⟩ tool, which will be used to represent the side of length 1 in the tiles. Next, construct the straight line *AB*, using [Lines]Line ⟨⟩, then the perpendicular to *AB* through a using [Constructions]Perpendicular line ⟨⟩, and the circle through centre A using [Curves]Circle ⟨⟩ and selecting first the centre *A* then the point on the circumference *B*. Finally, construct the point of intersection *C* of the perpendicular to *AB* and the circle. Select the point of intersection which is «above» *AB*, using [Points]Point ⟨•⟩.

We must now divide the circle into 10 equal sectors. Construct the point *B′* diametrically opposite to *A*, centre *B*, and *A′* diametrically opposite to *B*, centre *A*. (Use the [Transformations]Point Symmetry ⟨•••⟩ tool; by first selecting the point to be transformed, then the centre of symmetry.) If *A* is considered as the origin on line *AB*, then *B* is at 1, *B′* at 2, and *A′* at -1. We also need *A″*, the midpoint of *AA′*, using [Constructions]Midpoint ⟨•-•⟩. *A″* is at -1/2 using the above conventions.

Now construct the circle, centre *A″*, which passes through *C*. This circle cuts the line *AB* in two points: *P* to the left of *A*, and *Q* to the right of *A*. *P* and *Q* are on *AB* at - Ø and Ø-1 respectively. The perpendiculars to *AB*, through *P* and *Q* cut the circle, centre *A*, radius *AB′* in four points: the vertices of a regular pentagon, whose fifth vertex is at *B′*. Using symmetry, the vertices of the decagon can be completed as shown below. From this, the angle of Ø = 36º and length Ø = $(1+\sqrt{5})/2$ are constructed. These dimensions are intimately linked to the regular pentagon.

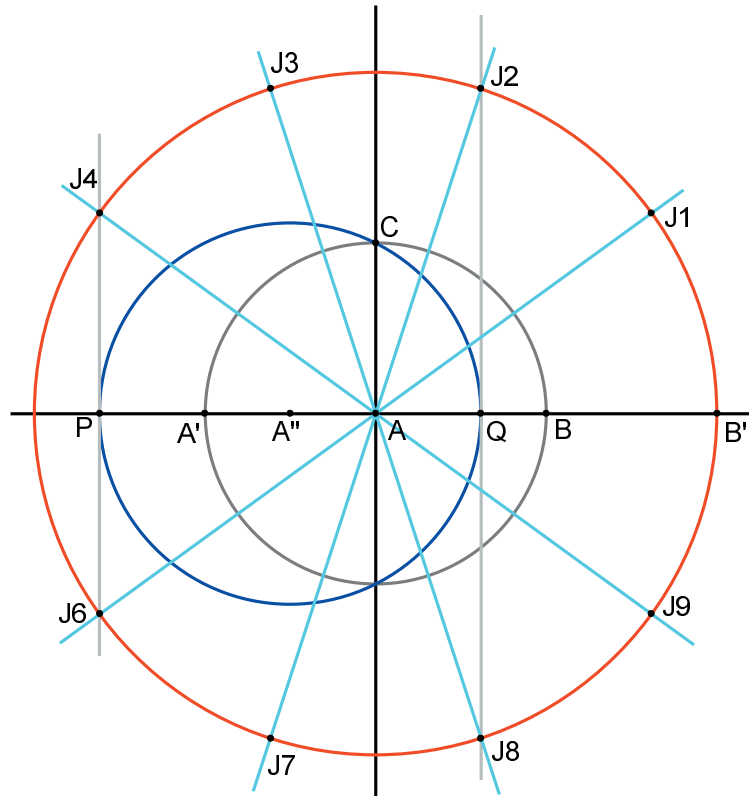**Figure 2** - *Division of the circle into 10 equal sectors*

Draw the circle, centre *A*, through *P*. The radius of this circle is then Ø. Now use [Attributes]Hide/Show ⬜ to hide all those elements of construction which do not appear in the figure below. The vertices of the regular decagon inscribed in the circle, radius Ø, are labelled *R*, 1, 2, 3, 4, *P*, 6, 7, 8, 9.
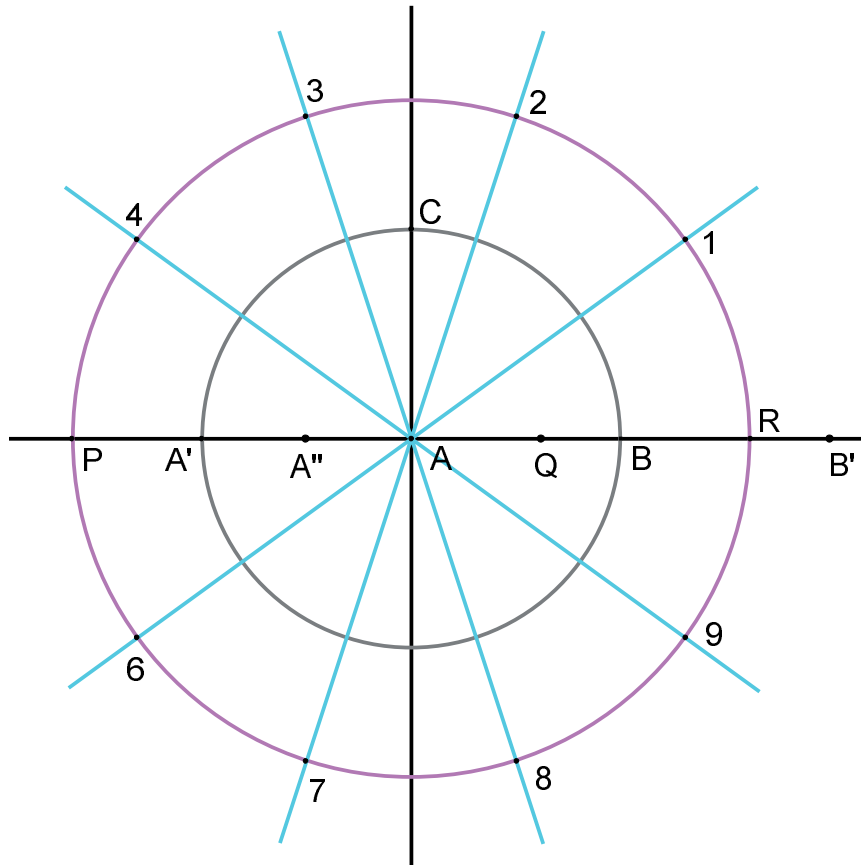
Figure 3 *- The subdivision into equal segments is transfered to the circle radius Ø, and the obsolete elements of construction are hidden.*

The construction can be continued by taking note of the following figure. The line segments linking *P* to points 2 and 8 are constructed using [Lines]Segment ✏️, then the two quadrilaterals are drawn with [Lines]Polygon ✏️. Next, the circles from which the arcs are to be selected are drawn using [Curves]Circle ✏️, and finally the arcs are drawn using [Curves]Arc ✏️. An arc of a circle is defined by three points: one extremity, an intermediate point, and the second extremity. The points used to define the arcs can now be hidden so that they do not appear when the macro is used. The appearance of the arcs and the quadrilaterals can be changed, using [Attributes]Thickness ✏️, and [Attributes]Colour ✏️.
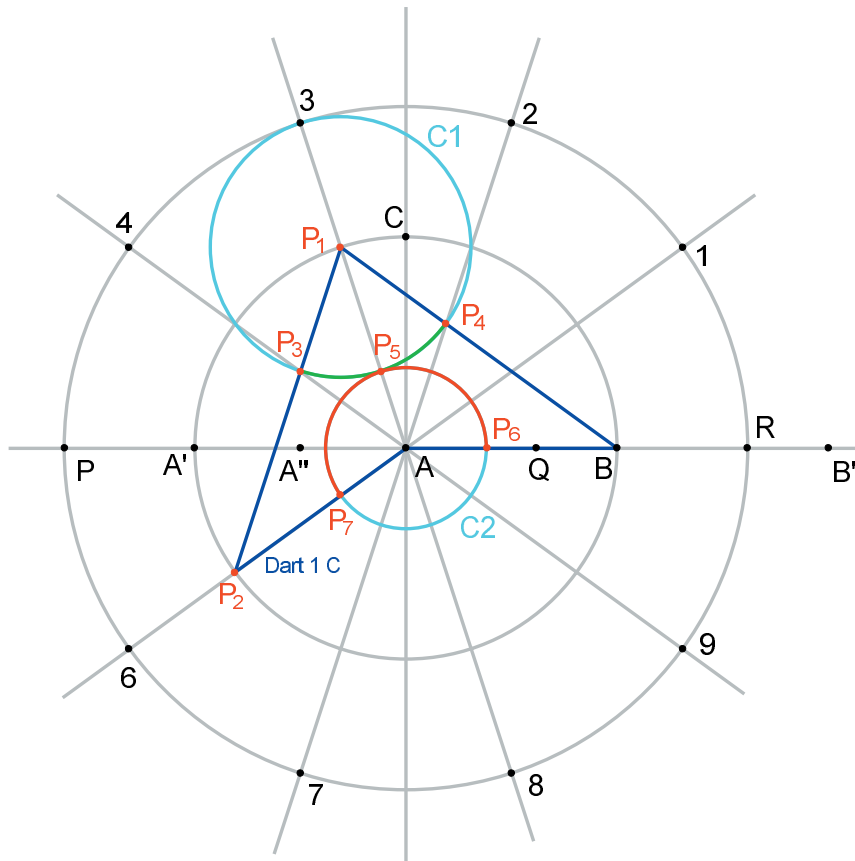
Figure 4 - Construction of the two tiles with their coloured arcs.

We can now create two of the macros. Activate the [Macros]Initial Objects ⟨X→⟩ tool, and select *B* then *A*. The order of selection of objects of the same type is important; they must be selected in the same order when the macro is used. Now activate the [Macros]Final Objects ⟨→Y⟩ tool and select the Dart and its two arcs. Finally the macro is finished by activating [Macros]Define Macro ⟨X→Y⟩. Name it Dart 1 L. As part of the process of defining a macro, an icon can be designed for the tool. It can also be named, a few lines of text can be entered for the Help window, a name can be given to the first object created, and the macro can be protected with a password. (Passwords are useful in a class situation for decoding «Black Box» tools.)

Once the macro has been defined, a new tool appears in the [Macros]toolbox. We will test our new macro. Activate the new [Macros]Dart 1 L tool, and select two new points: *U* and *V*. A new Dart, based on U and V, appears immediately.
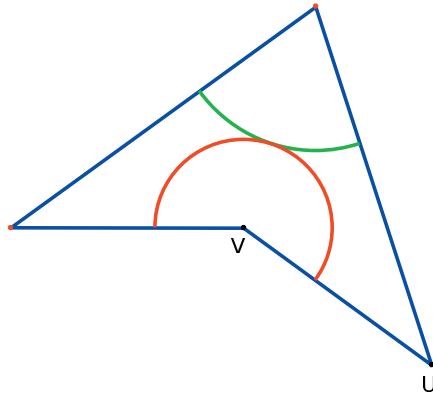
Figure 5 - *Application of the* *macro Dart I L* *to the two new points:* U *and* V

Using the earlier construction, we can define the macro `Kite 1 L` similarly. Using these two macros, we can start to construct the «Sun» tessellation, which has the same symmetry group as the regular pentagon.
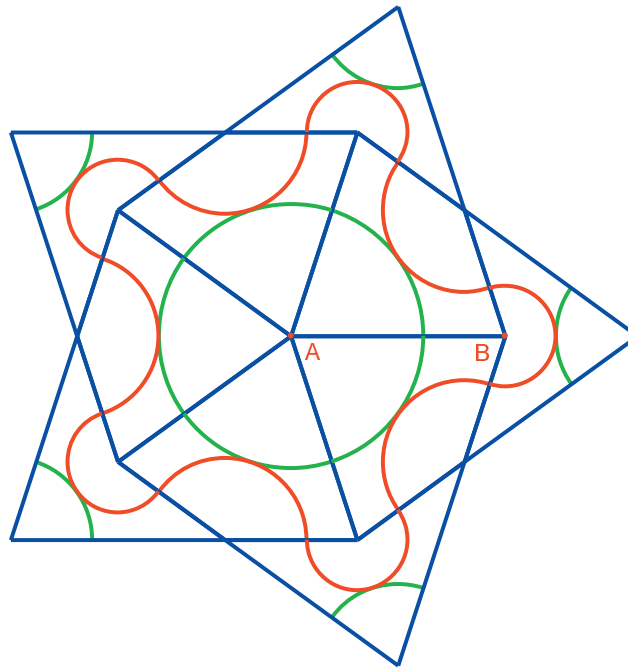

Figure 6 - *The start of the «Sun» tessellation, constructed using our two macros.*

Exercise 1 - Define the other six macros listed in the text above and continue the «Sun» tessellation. Draw the «Star» tessellation, whose centre is created from five Dart tiles pointing towards the central vertex.

Exercise 2 - List the 7 possible configurations of the Penrose tiles around a vertex.